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set $(i, j, \ldots\ldots, n+1)$, there are $\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} m \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} m-n-1 \\ r-s-1 \end{bmatrix}$ of the type s $s=[0, 1, \ldots\ldots, (r-1)]$. If the array have M rows and N columns the total number of type s is $\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} N \\ r \end{bmatrix}\begin{bmatrix} M \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} M-n-1 \\ r-s-1 \end{bmatrix}$. These relations between $(n \times n)$ and $(r \times r)$ determinants, $n > r$, can be reduced to sums of products of determinants of $(n-r)^2$ elements and the relations between determinants of the same order r added to products of $(n-r)$th order determinants and identities between determinants of order r which vanish in the determinants.

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SEVERAL FUNDAMENTAL THEOREMS IN GROUP THEORY.

By DR. G. A. MILLER.

The following theorems are readily deduced from known theorems. They appear to be of sufficient importance to be explicitly stated in view of their numerous elementary applications.

Theorem I. *If a group contains more than one cyclic subgroup of order* $k=p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots\ldots\ldots, p_\lambda^{\alpha_\lambda}$, *where* $p_1, p_2, \ldots\ldots\ldots, p_\lambda$ *are distinct prime numbers such that* $p_1 < p_2 < \ldots\ldots\ldots < p_\lambda$, *then it contains at least* p_1 *such subgroups. If the number of these subgroups is exactly* p_1 *the value of* α_1 *exceeds unity.*

This theorem may readily be derived from the known theorems that the number of cyclic subgroups of order p^α, $\alpha > 1$ and $p > 2$, in a non-cyclic group of order p^m is always divisible by p, and that the total number of subgroups of order p^α in such a group is of the form $1+mp$. Suppose that a group G contains more than one cyclic subgroup of order k but that the number of these subgroups is less than p_1+1. Each of these subgroups is transformed into itself by every other one, since an operator of order p^β transforms things in multiples of p when it does not transform them into themselves.

From this it follows that two of these cyclic subgroups generate a group which is the direct product of its Sylow subgroup. Since these Sylow subgroups cannot all be cyclic it is proved that this direct product contains at least p_1 cyclic subgroups of order k, and hence G contains at least p_1 cyclic subgroups of this order. From the fact that a group of order p^m cannot contain exactly p cyclic subgroups of order p^α when $\alpha=1$, it follows directly that $\alpha_1 > 1$ whenever G contains exactly p_1 cyclic subgroups of order k.

As an interesting particular consequence of this theorem we have that there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number. Moreover, it is easy to construct groups containing exactly two cyclic subgroups of any arbitrary order which is divisible by 4. For instance, there is an infinite number of groups containing just two cyclic subgroups of order twelve, but there is not a single group which contains just two cyclic subgroups of order six. Similar remarks evidently apply to the other prime numbers.

Theorem II. *If a group contains exactly p cyclic subgroups of order k, then it contains only one subgroup of order p^{α_λ}, $\lambda > 1$, and hence its Sylow subgroups whose orders are divisible by p_λ are cyclic.*

If these Sylow subgroups were not cyclic G would contain at least p_λ subgroups of order $p_\lambda^{\alpha_\lambda}$. This is impossible since each of these subgroups would transform into itself each of the p_1 cyclic subgroups of order k, and hence it would also transform each of the operators of such a subgroup, with a possible exception of those whose orders are divisible by p_λ, into itself. From this it follows directly that there would be more than p_1 cyclic subgroups of order k in G. In the same manner it may be observed that if the order of G is divisible by any prime which exceeds p_λ all its operators whose orders are powers of this prime are commutative with each operator in the p_1 cyclic subgroups of order k.

Theorem III. *A necessary and sufficient condition that a group is the direct product of its Sylow subgroups is that the nth power of each of its operators is contained in every subgroup of index n, for every possible value of n.*

That this condition is sufficient follows directly from the fact that if G would contain $1+kp$ Sylow subgroups of order p^m all the operators which would transform such a subgroup into itself would constitute a subgroup of index $1+kp$. As the latter would not contain all the operators whose orders are powers of p it could not contain the $1+kp$ power of every operator of G.

That the condition is also necessary follows from the fact that every subgroup of order p^α which is contained in a group of order p^m is itself invariant in a subgroup of order $p^{\alpha+1}$ and hence it includes the pth power of all the operators of the latter subgroup. Similarly, the latter subgroup contains the pth power of all the operators of a subgroup of order $p^{\alpha+2}$ and hence the given subgroup of order p^α contains the p^2 power of all the operators in this subgroup of order $p^{\alpha+2}$. This reasoning can clearly be continued until we arrive at the entire group of order p^m.

It is well known that another necessary and sufficient condition that G is the direct product of its Sylow subgroups is that we arrive at the identity by finding the group of cogredient isomorphisms of G, and then finding the group of cogredient isomorphisms of this group of cogredient isomorphisms, and then the third group of cogredient isomorphisms, etc. A group which is its own group of cogredient isomorphisms contains no invariant operator besides the identity, and vice versa.